Some isoperimetric inequalities with application to the Stekloff problem

by

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Abstract. In this paper we establish isoperimetric inequalities for the product of some moments of inertia. As an application, we obtain an isoperimetric inequality for the product of the N first nonzero eigenvalues of the Stekloff problem in $\mathbb{R}^N$.

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1 Introduction

Let $\mathbf{x} := (x_1, ..., x_N)$ be cartesian coordinates in $\mathbb{R}^N$, $N \geq 2$. Let $J_k(\Omega)$ be the moment of inertia of $\Omega \subset \mathbb{R}^N$ with respect to the plane $x_k = 0$, defined as

$$(1.1) \qquad J_k(\Omega) := \int_\Omega {x_k}^2 \, d\mathbf{x} \ , \quad k = 1, ..., N \ .$$

By summation over k, we obtain the polar moment of inertia of Ω with respect to the origin denoted by $J_0(\Omega)$,

$$(1.2) \qquad J_0(\Omega) := \sum_{k=1}^{N} J_k(\Omega) \ .$$

Clearly, $J_0(\Omega)$ depends of the position of the origin. In fact, $J_0(\Omega)$ is smallest when the origin coincides with the center of mass of Ω, i.e. when we have

$$(1.3) \qquad \int_\Omega x_k \, d\mathbf{x} = 0 \ , \quad k = 1, ..., N \ .$$

The following isoperimetric property is well known [3, 14] :

Theorem 1. *Among all domains Ω of prescribed N-volume, the ball Ω^* centered at the origin has the smallest polar moment of inertia, i.e. we have the isoperimetric inequality*

$$(1.4) \qquad J_0(\Omega) \geq J_0(\Omega^*) \ , \quad \Omega \in \mathcal{O} \ ,$$

where $\mathcal{O}$ is the class of all bounded domains of prescribed N-volume.

Let $I_k(\Omega)$ be the moment of inertia of $\partial\Omega$ with respect to the plane $x_k = 0$ defined as

$$(1.5) \qquad I_k(\Omega) := \int_{\partial\Omega} {x_k}^2 \, ds \ , \quad k = 1, 2, ..., N \ ,$$

where ds is the area element of the boundary $\partial\Omega$ of Ω. By summation over k, we obtain the polar moment of inertia $I_0(\Omega)$ of $\partial\Omega$ with respect to the origin

$$(1.6) \qquad I_0(\Omega) := \sum_{k=1}^{N} I_k(\Omega) \ .$$

$I_0(\Omega)$ is smallest when the origin coincides with the center of mass of $\partial\Omega$, i.e. when we have

$$\int_{\partial\Omega} x_k \, ds = 0 \ , \quad k = 1, ..., N \ . \tag{1.7}$$

Betta et al. [1] have derived the following isoperimetric property.

Theorem 2.

$$I_0(\Omega) \geq I_0(\Omega^*) \ , \quad \Omega \in \mathcal{O} \ , \tag{1.8}$$

with equality if and only if Ω *coincides with* Ω^*.

The quantities of interest in this paper are

$$J(\Omega) := \prod_{k=1}^{N} J_k(\Omega) \ , \tag{1.9}$$

and

$$I(\Omega) := \prod_{k=1}^{N} I_k(\Omega) \ . \tag{1.10}$$

We have several motivations to look at this products. Our main motivation was to extend a classical result of Hersch, Payne and Schiffer, see Theorem 5 below.
Another clear motivation is the following. Let us consider the inertia matrix of a body Ω (we can do the same for its boundary $\partial\Omega$), i.e. the matrix $\mathbf{M}$ whose general term is

$$\mathbf{M}_{ij} = \int_{\Omega} x_i x_j \, d\mathbf{x} \ , \quad i, j = 1, ..., N \ .$$

The most classical invariants of this matrix are its trace and its determinant and it is natural (for example for a mechanical point of view) to ask for the domains which minimize these invariants among all domains of prescribed N-volume. For the trace, the answer is given in the Theorem 1. For the determinant, we will see in section 2, that the ellipsoids symmetric with respect to the planes $x_k = 0$ are the minimizers of the determinant: it will be a simple consequence of the study of the product J.

A last motivation can be found in a paper of G. Polya, [12]. Indeed, in this paper, the author was able to get the following upper bound for the *torsional rigidity* $P(\Omega)$ of an elastic beam with cross section Ω:

$$J_0(\Omega)P(\Omega) \leq J(\Omega)$$

and, then was led to look for the minimizers of $J(\Omega)$ among plane domains of given area.

Let us now describe the content of this paper. In Section 2 we establish the following isoperimetric property.

Theorem 3.

$$J(\Omega) \geq J(E) \ , \tag{1.11}$$

valid for all domains $\Omega \in \mathcal{O}$, with equality for all ellipsoids E ($\in \mathcal{O}$) symmetric with respect to the planes $x_k = 0$, $k = 1, ..., N$.

We note that the isoperimetric inequality (1.11) also follows from Blaschke's great contribution to affine geometry [2].

In Section 3 we establish the following isoperimetric property.

Theorem 4.

$$I(\Omega) \geq I(\Omega^*) \ , \tag{1.12}$$

valid for all convex domains $\Omega \in \mathcal{O}$, with equality if and only if $\Omega = \Omega^*$.

As an application of (1.12), we establish in Section 4 an isoperimetric inequality for the product of the N first nonzero eigenvalues of the Stekloff problem in $\mathbb{R}^N$. This inequality generalizes to dimension N a previous two-dimensional result of Hersch, Payne and Schiffer, see [7], [8, Theorem 7.3.4]:

Theorem 5. *Let Ω be a convex domain in $\mathbb{R}^N$ and $0 = p_1(\Omega) < p_2(\Omega) \leq p_3(\Omega) \leq ...$ the eigenvalues of Ω for the Stekloff problem, see (4.1). Then, the following isoperimetric inequality holds*

$$\prod_{k=2}^{N+1} p_k(\Omega) \leq \prod_{k=2}^{N+1} p_k(\Omega^*) \tag{1.13}$$

with equality if and only if $\Omega = \Omega^*$.

In (1.12), the convexity of Ω may not be required. In fact we show in Section 5 that (1.12) remains valid for nonconvex domains Ω in $\mathbb{R}^2$.

2 The functional J and proof of (1.11)

The proof of (1.11) is based on the fact that $J(\Omega)$ is not affected by an affinity, i.e. when Ω is replaced by

$$\Omega' := \{ \ \mathbf{x}' := (t_1 x_1, ..., t_N x_N) \in \mathbb{R}^N \ | \ \mathbf{x} \in \Omega \ \} \ , \tag{2.1}$$

where t_k are N arbitrary positive constants such that

$$\prod_{k=1}^{N} t_k = 1 \ . \tag{2.2}$$

We compute indeed

$$J_k(\Omega) = {t_k}^{-2} J_k(\Omega') \ , \quad k = 1, ..., N \ , \tag{2.3}$$

which implies

$$J(\Omega) = J(\Omega') \ . \tag{2.4}$$

With the particular choice

$${t_k}^2 := (J(\Omega))^{\frac{1}{N}} J_k^{-1}(\Omega) \ , \quad k = 1, ..., N \tag{2.5}$$

in (2.3), we obtain

$$J_k(\Omega') = (J(\Omega))^{\frac{1}{N}} \ , \quad k = 1, ..., N \ , \tag{2.6}$$

i.e. the values of $J_k(\Omega')$ are independent of k. This shows that we have

$$\min_{\Omega \in \mathcal{O}} J(\Omega) = \min_{\Omega' \in \mathcal{O}'} J(\Omega') \ , \tag{2.7}$$

where $\mathcal{O}'$ ($\subset \mathcal{O}$) is the class of all domains Ω' of prescribed N-volume such that $J_1(\Omega') = ... = J_N(\Omega')$. Moreover we have by (2.4), (2.6),

$$J_0(\Omega') := \sum_{k=1}^{N} J_k(\Omega') = N(J(\Omega))^{\frac{1}{N}} = N(J(\Omega'))^{\frac{1}{N}} \ , \tag{2.8}$$

from which we obtain

$$J(\Omega') = \left(\frac{1}{N} J_0(\Omega') \right)^N \ . \tag{2.9}$$

Combining (2.7) and (2.9), and making use of (1.4), we are led to

$$(2.10)\qquad \begin{aligned}\min_{\Omega\in\mathcal{O}} J(\Omega) &= \min_{\Omega'\in\mathcal{O}'}\left\{\frac{1}{N}J_0(\Omega')\right\}^N = \frac{1}{N^N}\left(\min_{\Omega'\in\mathcal{O}'}\{J_0(\Omega')\}\right)^N \\ &= \frac{1}{N^N}\left(J_0(\Omega^*)\right)^N = J(\Omega^*) = J(E)\ ,\end{aligned}$$

which is the desired result.

We give now an application to the minimization of the determinant of the inertia matrix. We assume that the origin O is at the center of mass of the domains we consider.

Corollary 1. *Let $\mathbf{M}(\Omega)$ be the inertia matrix of the domain Ω i.e. the matrix whose general term is*

$$\mathbf{M}_{ij} = \int_\Omega x_i x_j\, d\mathbf{x}\ ,\quad i,j = 1,...,N$$

and let $D(\Omega)$ be its determinant. Then,

$$(2.10)\qquad D(\Omega) \geq D(E)$$

valid for all domains $\Omega \in \mathcal{O}$, with equality for all ellipsoids E ($\in \mathcal{O}$) symmetric with respect to the planes $x_k = 0$, $k = 1,...,N$.

Indeed, since $\mathbf{M}(\Omega)$ is symmetric, there exists an orthogonal matrix $T \in O^+(N)$ and a diagonal matrix Δ such that $\mathbf{M}(\Omega) = T^T \Delta T$. Actually, Δ is the inertia matrix of the domain $(T(\Omega)$ obtained from Ω by some rotation. Now, the determinant being invariant through such a similarity transformation, we have according to (1.11):

$$(2.11)\qquad D(\Omega) = D(T(\Omega)) = J(T(\Omega)) \geq J(E) = D(E)$$

the last equality in (2.11) coming from the fact that E is symmetric with respect to each plane of coordinates, so its inertia matrix is diagonal. This proves the desired result.

3 The functional I and proof of (1.12)

We assume in this section that Ω is convex and choose the origin at the center of mass of $\partial\Omega$. We introduce the family of parallel domains

$$(3.1)\qquad \begin{aligned}\Omega_h &:= \Omega + B_h(0) \\ &= \{\ \mathbf{z} = \mathbf{x} + \mathbf{y} \in \mathbb{R}^N \mid \mathbf{x} \in \Omega\ ,\ \mathbf{y} \in B_h(0)\ \}\ ,\end{aligned}$$

where $B_h(0)$ is the N-ball of radius $h > 0$ centered at the origin. It follows from the Brunn-Minkowski theory that the function $f(h) := |\Omega_h|^{1/N}$ is concave with respect to the parameter h. H. Minkowski made use of this basic property to derive the famous classical isoperimetric geometric inequality for convex bodies. Since our approach will be patterned after his argument, we indicate here briefly Minkowski's method. The concavity of $f(h)$ implies that $f'(h)$ is a monotone decreasing function. This leads to the inequality

$$f'(h) = \frac{1}{N} |\Omega_h|^{\frac{1-N}{N}} |\Omega_h|' = \frac{1}{N} |\Omega_h|^{\frac{1-N}{N}} |\partial\Omega_h| \geq C \ , \tag{3.2}$$

where C has to coincide with the value of $f'(h)$ for a ball since Ω_h approaches a large ball as h increases to infinity. Evaluated at $h = 0$, (3.2) leads to the well known isoperimetric geometric inequality

$$|\partial\Omega|^N \geq N^N \omega_N |\Omega|^{N-1} \ , \tag{3.3}$$

where $\omega_N := \pi^{N/2} / \Gamma\left(\frac{N}{2} + 1\right)$ is the volume of the unit ball in $\mathbb{R}^N$. We refer the reader to the basic books of Bonnesen-Fenchel [3] and Hadwiger [5, 6] for details. The proof of (1.12) makes use of the following two lemmas.

Lemma 1. *With the notations of Section 1, we have*

$$J_k(\Omega_h) = J_k(\Omega) + hI_k(\Omega) + \sum_{j=2}^{N+2} c_j h^j \ , \quad h \geq 0 \ , \tag{3.4}$$

where the coefficients c_j are some geometric quantities associated to $\partial\Omega$.

As a consequence of (3.4), we have

$$J_k{}'(h)\big|_{h=0} = I_k(\Omega) \ . \tag{3.5}$$

For the proof of Lemma 1, we evaluate $\int_{\Omega_h \setminus \Omega} x_k^2 \, d\mathbf{x} = J_k(\Omega_h) - J_k(\Omega)$. The computation of this integral will be easy if we introduce normal coordinates $\mathbf{s} := (s_1, ..., s_N)$ such that

$$\Omega_h \setminus \Omega = \{\mathbf{x}(\mathbf{s}) = \mathbf{r}(s_1, ..., s_{N-1}) + s_N \mathbf{n}(s_1, ..., s_{N-1}) \ , \quad \mathbf{s} \in B \times [0, h]\} \ , \tag{3.6}$$

where $\mathbf{r}(s_1, ..., s_{N-1})$, $(s_1, ..., s_{N-1}) \in B$, is a parametric representation of $\partial\Omega$, and $\mathbf{n}(s_1, ..., s_{N-1})$ is the unit normal vector of $\partial\Omega$. In terms of the new

variables $\mathbf{s}$, the volume element of $\mathbb{R}^N$ is $dV = \Delta\, ds_1 ... ds_N$, with

$$\begin{aligned} (3.7) \qquad \Delta &= \det \left| \frac{\partial x_k}{\partial s_j} \right| \\ &= \det \left| \frac{\partial \mathbf{r}}{\partial s_1} + s_N \frac{\partial \mathbf{n}}{\partial s_1} \, , \, ... \, , \, \frac{\partial \mathbf{r}}{\partial s_{N-1}} + s_N \frac{\partial \mathbf{n}}{\partial s_{N-1}} \, , \, \mathbf{n} \right| \; . \end{aligned}$$

Δ is obviously a polynomial in s_N of degree $(N-1)$. We have

$$\Delta = \Delta_0 + \sum_{j=1}^{N-1} \widetilde{c_j} \, (s_N)^j \quad , \tag{3.8}$$

where Δ_0 is Δ evaluated on $\partial\Omega$. We then obtain

$$\begin{aligned} \int_{\Omega_h \setminus \Omega} x_k^2 \, d\mathbf{x} &= \int_{B \times [0,h]} (x_k^2 \Delta_0 + \sum_{j=1}^{N-1} x_k^2 \widetilde{c_j} \left(s_N \right)^j) \; ds_1 \ldots ds_N \\ &= h I_k(\Omega) + \sum_{j=2}^{N+2} c_j h^j \; , \quad h \geq 0 \; , \end{aligned}$$

which is the desired result (3.4).

The next lemma follows from an extension of the Brunn-Minkowski inequality established by H. Knothe [11].

Lemma 2. *The functions* $g(h) := (J_k(\Omega_h))^{\frac{1}{N+2}}$ *are concave for* $h \geq 0$, $k = 1, ..., N$.

As a direct consequence of Lemmas 1 and 2, we have in analogy to (3.2)

$$g'(h) = \frac{1}{N+2} \left(J_k\left(\Omega_h\right)\right)^{-\frac{N+1}{N+2}} I_k(\Omega_h) \geq C \; , \quad k = 1, ..., N \; , \tag{3.9}$$

where C has to coincide with the value of $g'(h)$ for a N-ball. Using $J_k(B_R) = \frac{R^{N+2}\omega_N}{N+2}$ for the ball of radius R, we get $C = \left(\frac{\omega_N}{N+2}\right)^{1/(N+2)}$. Evaluated at $h = 0$, (3.9) leads to the following isoperimetric inequalities

$$(I_k(\Omega))^{N+2} \geq (N+2)^{N+1} \omega_N (J_k(\Omega))^{N+1} \; , \quad k = 1, ..., N \; , \tag{3.10}$$

with equality if and only if $\Omega = \Omega^*$. Making use of (3.10) and (1.11), we obtain

$$\begin{aligned} (I(\Omega))^{N+2} &\geq (N+2)^{N(N+1)}(\omega_N)^N (J(\Omega))^{N+1} \\ &\geq (N+2)^{N(N+1)}(\omega_N)^N (J(\Omega^*))^{N+1} = (I(\Omega^*))^{N+2} , \end{aligned}$$

with equality if and only if $\Omega = \Omega^*$. This achieves the proof of inequality (1.12) for convex domains Ω.

4 Application to the Stekloff problem

In this section we consider the Stekloff eigenvalue problem defined in a bounded convex domain Ω in $\mathbb{R}^N$, $N \geq 2$.

$$(4.1) \qquad \begin{cases} \Delta u = 0 & , \quad \mathbf{x} := (x_1, ..., x_N) \in \mathbb{R}^N , \\ \dfrac{\partial u}{\partial n} = pu & , \quad \mathbf{x} \in \partial\Omega . \end{cases}$$

In (4.1), $\frac{\partial u}{\partial n}$ is the exterior normal derivative of u on $\partial\Omega$. It is well known [15] that there are infinitely many eigenvalues $0 = p_1 < p_2 \leq p_3 \leq ...$ for which (4.1) has nontrivial solutions, also called eigenfunctions, and denoted by $u_1 (= \text{const.}), u_2, u_3, ...$ Let Σ_k be the class of functions defined as

$$(4.2) \quad \Sigma_k := \left\{ v \in H^1(\Omega) \; , \; \int_{\partial\Omega} v u_j \, ds = 0 \; , \; j = 1, ..., k-1 \right\} ,$$

where $H^1(\Omega)$ is the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives are in $L^2(\Omega)$. Let $R[v]$ be the the Rayleigh quotient associated to the problem (4.1) defined as

$$(4.3) \qquad R[v] := \frac{\int_\Omega |\nabla v|^2 \, d\mathbf{x}}{\int_{\partial\Omega} v^2 \, ds} .$$

It is well known that the eigenvalue p_k has the following variational characterization [7, 8]

$$(4.4) \qquad p_k = \min_{v \in \Sigma_k} R[v] \; , \quad k = 2, 3, 4, ...$$

Unfortunately, (4.4) is of little practical use for estimating p_k, since it requires the knowledge of the eigenfunctions u_j, $j = 1, ..., k-1$. The following variational characterization due to H. Poincaré overcomes this difficulty. Let

$v_k(\not\equiv 0) \in H^1(\Omega),\ k = 1, ..., n$ be n linearly independent functions satisfying the conditions

$$(4.5) \qquad \int_{\partial\Omega} v_k\, ds = 0 \ , \quad k = 1, ..., n \ .$$

Let L_n be the linear space generated by $v_k,\ k = 1, ..., n$. The Rayleigh quotient of $v := \sum_{k=1}^n c_k v_k$ is the ratio of two quadratic forms of the n variables $c_1, ..., c_n$. We have

$$(4.6) \qquad R[v] := \frac{\int_\Omega |\nabla v|^2\, d\mathbf{x}}{\int_{\partial\Omega} v^2\, ds} = \frac{\sum_{i,j=1}^n a_{ij} c_i c_j}{\sum_{i,j=1}^n b_{ij} c_i c_j} \ ,$$

with

$$(4.7) \qquad a_{ij} \ := \ \int_\Omega \nabla v_i \nabla v_j\, d\mathbf{x} \ ,$$

$$(4.8) \qquad b_{ij} \ := \ \int_{\partial\Omega} v_i v_j\, ds \ .$$

Note that the matrices $A := (a_{ij}), B := (b_{ij})$ are positive definite. Let $0 < p_2' \le p_3' \le ... \le p_{n+1}'$ be the n roots of the characteristic equation

$$(4.9) \qquad \det |A - pB| = 0 \ .$$

Poincaré's variational principle [13] asserts that

$$(4.10) \qquad p_k \le p_k' \ , \quad k = 2, ..., n+1 \ .$$

By means of a translation followed by an appropriate rotation, the following conditions will we satisfied

$$(4.11) \qquad \int_{\partial\Omega} x_k\, ds \ = \ 0 \ , \quad k = 1, ..., N \ ,$$

$$(4.12) \qquad \int_{\partial\Omega} x_k x_j\, ds \ = \ 0 \ , \quad k \ne j \ .$$

The N functions defined as

$$(4.13) \qquad v_k := x_k (I_k(\Omega))^{-1/2} \ , \quad k = 1, ..., N$$

are admissible for the Poincaré principle. We then compute with the notation of Section 1

$$(4.14) \qquad A \ = \ |\Omega| \operatorname{diag}\big(I_1^{-1}(\Omega), ..., I_N^{-1}(\Omega) \big) \ ,$$

$$(4.15) \qquad B \ = \ \operatorname{diag}\big(1, ..., 1 \big) \ .$$

The N roots of the characteristic equation (4.9) are then $|\Omega|I_k^{-1}(\Omega)$, $k = 1, ..., N$. We then obtain from (4.10) with $n = N$

$$\begin{aligned}(4.16) \qquad \prod_{k=2}^{N+1} p_k(\Omega) \quad &\leq \quad \prod_{k=2}^{N+1} p'_k = |\Omega|^N I^{-1}(\Omega) \\ &\leq \quad |\Omega|^N I^{-1}(\Omega^*) = \frac{\omega_N}{|\Omega|} = \prod_{k=2}^{N+1} p_k(\Omega^*) \ .\end{aligned}$$

In (4.16), we have used the isoperimetric inequality (1.12) and the fact that x_k, $k = 1, ..., N$, are the N first nonzero eigenvalues of Ω^*.
Note that (4.16) is an improvement (for convex Ω!) of the following inequality

$$(4.17) \qquad \sum_{k=2}^{N+1} \frac{1}{p_k(\Omega)} \geq \sum_{k=2}^{N+1} \frac{1}{p_k(\Omega^*)} \ ,$$

obtained by Brock [4].

5 Appendix

Since our proof of (1.12) is valid only for convex Ω, we indicate in this section a proof inspired by an old paper of A. Hurwitz [10] that does not require convexity of $\Omega \subset \mathbb{R}^2$. Let L be the length of $\partial\Omega$ and s be the arc length on $\partial\Omega$. Consider the following parametric representation of $\partial\Omega$:

$$(5.1) \qquad (x(\sigma), y(\sigma)) \ , \quad \sigma := \frac{2\pi}{L} s \quad \in \ [0, 2\pi] \ .$$

Clearly $x(\sigma)$ and $y(\sigma)$ are 2π-periodic functions of σ whose associated Fourier series are of the form

$$(5.2) \qquad \begin{cases} x(\sigma) = \dfrac{1}{2}a_0 + \displaystyle\sum_{k=1}^{\infty}(a_k \cos k\sigma + a'_k \sin k\sigma) \ , \\ y(\sigma) = \dfrac{1}{2}b_0 + \displaystyle\sum_{k=1}^{\infty}(b_k \cos k\sigma + b'_k \sin k\sigma) \ . \end{cases}$$

The Fourier coefficients a_k, a'_k, b_k, b'_k have to be determined in order to

minimize $I(\Omega) = I_1(\Omega) I_2(\Omega)$. ¿From (5.2) and Parseval's identity, we obtain

$$(5.3)\qquad \frac{L^2}{2\pi} = \frac{1}{\pi}\int_0^{2\pi} \left\{ \left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 \right\} d\sigma = \sum_{k=1}^{\infty} k^2 (a_k^2 + {a'_k}^2 + b_k^2 + {b'_k}^2) \;,$$

$$(5.4)\qquad |\Omega| = \int_0^{\pi} x(\sigma)\frac{dy}{d\sigma}\, d\sigma = \pi \sum_{k=1}^{\infty} k(a_k b'_k - a'_k b_k) \;,$$

$$(5.5)\qquad I_1(\Omega) = \int_{\partial\Omega} x^2(s)\, ds = \frac{L}{2\pi}\int_0^{2\pi} x^2(\sigma)\, d\sigma = \frac{L}{2}\left\{\frac{1}{2}{a_0}^2 + a^2\right\} \;,$$

$$(5.6)\qquad I_2(\Omega) = \int_{\partial\Omega} y^2(s)\, ds = \frac{L}{2\pi}\int_0^{2\pi} y^2(\sigma)\, d\sigma = \frac{L}{2}\left\{\frac{1}{2}{b_0}^2 + b^2\right\} \;,$$

with

$$(5.7)\qquad a^2 := \sum_{k=1}^{\infty} ({a_k}^2 + {a'_k}^2), \quad b^2 := \sum_{k=1}^{\infty} ({b_k}^2 + {b'_k}^2) \;.$$

Clearly, we must choose $a_0 = b_0 = 0$ since L and $|\Omega|$ are independent of $a_0,\ b_0$. The other Fourier coefficients may be determined using Lagrange's method, consisting in finding the critical points of the Lagrange function defined as

$$(5.8)\qquad F(a, a', b, b') := 4I(\Omega) + \frac{\lambda}{\pi}|\Omega| = L^2 a^2 b^2 - \lambda \sum_{k=1}^{\infty} (a_k b'_k - a'_k b_k) \;,$$

where λ is a multiplier. This leads to the following system of equations

$$(5.9)\qquad \frac{\partial F}{\partial a_k} = 4\pi^2 k^2 a^2 b^2 a_k + 2L^2 b^2 a_k - \lambda k b'_k = 0 \;,$$

$$(5.10)\qquad \frac{\partial F}{\partial a'_k} = 4\pi^2 k^2 a^2 b^2 a'_k + 2L^2 b^2 a'_k + \lambda k b_k = 0 \;,$$

$$(5.11)\qquad \frac{\partial F}{\partial b_k} = 4\pi^2 k^2 a^2 b^2 b_k + 2L^2 a^2 b_k + \lambda k a'_k = 0 \;,$$

$$(5.12)\qquad \frac{\partial F}{\partial b'_k} = 4\pi^2 k^2 a^2 b^2 b'_k + 2L^2 a^2 b'_k - \lambda k a_k = 0 \;.$$

¿From (5.9), (5.12), we obtain

$$a_k M_k = 0 \ , \tag{5.13}$$

with

$$M_k := 4a^2b^2(2\pi^2k^2a^2 + L^2)(2\pi^2k^2b^2 + L^2) - k^2\lambda^2 \ . \tag{5.14}$$

¿From (5.10), (5.11), we obtain

$$a'_k M_k = 0 \ . \tag{5.15}$$

We conclude that either $a_k = a'_k = 0$, $k = 1, 2, 3, ...$, which is absurd, or that

$$\lambda^2 = 4a^2b^2k^{-2}(2\pi^2k^2a^2 + L^2)(2\pi^2k^2b^2 + L^2) = 4a^2b^2 f(k^2) \ , \tag{5.16}$$

with

$$f(t) := \frac{1}{t}(2\pi^2ta^2 + L^2)(2\pi^2tb^2 + L^2) \ , \quad t \geq 1 \ . \tag{5.17}$$

Since $f(t)$ is convex for $t \geq 1$, equation (5.16) can be satisfied for at most two positive integers $k_1 \leq k_2$. We then conclude that the parametric representation of $\partial\Omega^*$ is of the form

$$\begin{cases} x(\sigma) = a_{k_1}\cos(k_1\sigma) + a'_{k_1}\sin(k_1\sigma) + a_{k_2}\cos(k_2\sigma) + a'_{k_2}\sin(k_2\sigma) \ , \\ \\ y(\sigma) = b_{k_1}\cos(k_1\sigma) + b'_{k_1}\sin(k_1\sigma) + b_{k_2}\cos(k_2\sigma) + b'_{k_2}\sin(k_2\sigma) \ . \end{cases} \tag{5.18}$$

Further restrictions on the Fourier coefficients a_{k_j}, a'_{k_j}, b_{k_j}, b'_{k_j}, $j = 1, 2$, are imposed by the condition

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 = \left(\frac{L}{2\pi}\right)^2 = \text{const.} \tag{5.19}$$

¿From (5.18), we compute

$$\begin{aligned} &\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 = c_0 + c_1\cos(2k_1\sigma) + c_2\sin(2k_1\sigma) \\ &+c_3\cos(2k_2\sigma) + c_4\sin(2k_2\sigma) + c_5\cos(k_1 - k_2)\sigma \\ &+c_6\cos(k_1 + k_2)\sigma + c_7\sin(k_1 - k_2)\sigma + c_8\sin(k_1 + k_2)\sigma \ , \end{aligned} \tag{5.20}$$

with

$$(5.21)\qquad c_0 := \tfrac{1}{2}k_1^2(a_{k_1}^2 + a_{k_1}'^2 + b_{k_1}^2 + b_{k_1}'^2) + \tfrac{1}{2}k_2^2(a_{k_2}^2 + a_{k_2}'^2 + b_{k_2}^2 + b_{k_2}'^2)\ ,$$

$$(5.22)\qquad c_1 := \tfrac{1}{2}k_1^2(a_{k_1}'^2 + b_{k_1}'^2 - a_{k_1}^2 - b_{k_1}^2)\ ,$$

$$(5.23)\qquad c_2 := -k_1^2(a_{k_1}a_{k_1}' + b_{k_1}b_{k_1}')\ ,$$

$$(5.24)\qquad c_3 := \tfrac{1}{2}k_2^2(b_{k_2}'^2 + a_{k_2}'^2 - b_{k_2}^2 - a_{k_2}^2)\ ,$$

$$(5.25)\qquad c_4 := -{k_2}^2(a_{k_2}a_{k_2}' + b_{k_2}b_{k_2}')\ ,$$

$$(5.26)\qquad c_5 := k_1k_2(a_{k_1}a_{k_2} + b_{k_1}b_{k_2} + a_{k_1}'a_{k_2}' + b_{k_1}'b_{k_2}')\ ,$$

$$(5.27)\qquad c_6 := k_1k_2(-a_{k_1}a_{k_2} - b_{k_1}b_{k_2} + a_{k_1}'a_{k_2}' + b_{k_1}'b_{k_2}')\ ,$$

$$(5.28)\qquad c_7 := k_1k_2(a_{k_1}'a_{k_2} + b_{k_1}'b_{k_2} - a_{k_1}a_{k_2}' - b_{k_1}b_{k_2}')\ ,$$

$$(5.29)\qquad c_8 := -k_1k_2(a_{k_1}a_{k_2}' + b_{k_1}b_{k_2}' + a_{k_1}'a_{k_2} + b_{k_1}'b_{k_2})\ .$$

Suppose now that (5.19) is satisfied for two positive integers $k_1 \neq k_2$. Then we must have

$$(5.30)\qquad c_j = 0\ ,\quad j = 1, ..., 8 \ \text{ if } \ k_1 \neq 3k_2\ ,$$

or

$$(5.31)\qquad c_1 = c_2 = c_6 = c_8 = c_3 + c_5 = c_4 + c_7 = 0 \ \text{ if } \ k_1 = 3k_2\ .$$

A careful investigation of these two cases shows that we must have either $a_{k_1} = a_{k_1}' = b_{k_1} = b_{k_1}' = 0$, or $a_{k_2} = a_{k_2}' = b_{k_2} = b_{k_2}' = 0$. For the sake of brevity we omit the computational details to confirm this assertion. In any case the parametric representation of $\partial\Omega^*$ takes the following form

$$(5.32)\qquad \begin{cases} x(\sigma) = a_{k_0}\cos k_0\sigma + a_{k_0}'\sin k_0\sigma\ , \\ y(\sigma) = b_{k_0}\cos k_0\sigma + b_{k_0}'\sin k_0\sigma\ , \quad \sigma \in [0, 2\pi]\ , \end{cases}$$

for some positive integer k_0. But since $\partial\Omega^*$ makes only one loop around the origin, we must actually have $k_0 = 1$. We then obtain

$$(5.33) \quad \begin{cases} x(\sigma) = a_1 \cos\sigma + a'_1 \sin\sigma \\ \\ y(\sigma) = b_1 \cos\sigma + b'_1 \sin\sigma \ , \end{cases}$$

with

$$(5.34) \quad a_1^2 + b_1^2 = a_1'^2 + b_1'^2 \ , \quad a_1 a'_1 + b_1 b'_1 = 0 \ .$$

Finally (5.33), (5.34) lead to

$$(5.35) \quad x^2(\sigma) + y^2(\sigma) = a_1^2 + b_1^2 = \text{const.}$$

(5.35) shows that if there exists a minimizer $\widetilde{\Omega}$ of $I(\Omega)$, $\widetilde{\Omega} \in \mathcal{O}$, then it must be a disc centered at the origin. The existence of $\widetilde{\Omega}$ (among other similar results) will be established in a forthcoming paper of A. Henrot, [9].

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